

SIZE OF COMPONENTS OF A CUBE COLORING

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ABSTRACT. Suppose a d -dimensional lattice cube of size n^d is colored in several colors so that no face of its triangulation (subdivision of the standard partition into n^d small cubes) is colored in $m + 2$ colors. Then one color is used at least $f(d, m)n^{d-m}$ times.

1. INTRODUCTION

A theorem attributed to Lebesgue asserts that if a lattice cube of dimension d is colored in d colors then one of the colors has a connected component spanning two opposite facets of the cube. By the standard reasoning with nerves of coverings this means that the covering dimension of the d -dimensional cube is at least d .

There arises the following natural question: What happens if the number of colors is less than d ? In [4, 1] it was conjectured that the size of a monochromatic connected component has a lower bound of order n^{d-m} when $m + 1$ colors are used. For $m = d - 1$ this follows from the Lebesgue theorem and for $m = 1$ this conjecture is proved in [4].

Here we prove this conjecture in a slightly stronger form:

Theorem 1.1. *Let a d -dimensional cube Q be partitioned into n^d small cubes in the standard way and then the $(m + 1)$ -dimensional skeleton Q_m of this partition is subdivided to the triangulation T . Suppose the vertices of T (equivalently, vertices of Q_m) are colored in several colors so that no $(m + 1)$ -face $\sigma \in T$ is colored in $m + 2$ different colors. Then one of the colors is used on at least $f(d, m)n^{d-m}$ vertices of T .*

Remark 1.2. It is also sufficient to assume that every cubical face of Q_m of dimension $m + 1$ has at most $m + 1$ colors on its vertices. Such a point of view allows not to use any triangulation T in the statement of the theorem.

Remark 1.3. If a color c has several connected components then every component can be assumed to be a separate color. So we obtain a monochromatic connected set of at least $f(d, m)n^{d-m}$ vertices of T . By *adjacent* vertices we mean two vertices in a single $(m + 1)$ -face of T . This remark also remains valid if we define adjacent vertices as contained in a single cubical $(m + 1)$ -face of Q_m .

Remark 1.4. We do not establish any explicit values for $f(d, m)$. The reader may consult the paper [3] for this information.

A similar theorem was proved independently in [3]. In Section 3 we give some corollaries of Theorem 1.1 about coverings of a cube or a torus.

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2. PROOF OF THEOREM 1.1

Let us introduce some notation. Let $C = \{c_0, \dots, c_k\}$ be an ordered set of $k+1$ colors. For every oriented k -face of T we assign $+1$ if its vertices are colored in the colors of C in accordance with the orientation, -1 if the vertices are colored in the colors of C with opposite orientation, and 0 if the face is colored in other colors or some of the colors is used more than once. Thus we define a cochain $\chi(C) \in C^k(T)$ and observe the coboundary formula:

$$(2.1) \quad \delta\chi(C) = \sum_{c_{k+1}} \chi_{Cc_{k+1}},$$

where we sum over all the colors and by Cc_{k+1} we mean the concatenation of C and the new color c_{k+1} . If c_{k+1} coincides with a color in C then $\chi_{Cc_{k+1}}$ is assumed to be zero.

In the rest of the proof we consider $(k+1)$ -dimensional cubical subcomplexes $Q_k \subseteq Q_m$, which are $(k+1)$ -dimensional cubical skeleta of some $(d-m+k)$ -dimensional faces (big faces, not faces of a partition) of the cube Q , for $k = m, m-1, \dots, 0$.

In order to apply simplicial cochains to cubical chains we introduce the function $L: C_k(Q_m) \rightarrow C_k(T)$ that assigns to any k -face $\tau \in Q_m$ the sum of simplicial k -faces of T that partition τ with appropriate orientations. Obviously L commutes with the boundary map ∂ .

We are going to *balance* the complexes Q_k as follows:

Definition 2.1. For every k -face $\sigma \in Q_k$ we will assign a $(k+1)$ -chain $B(\sigma) \in C_{k+1}(T)$ so that for any $(k+1)$ -face $\tau \in Q_k$ and a set C of $k+1$ colors the following holds:

$$(2.2) \quad \chi_C(L(\partial\tau) + \partial B(\partial\tau)) = 0,$$

where we assume that B is linearly extended to k -chains of Q_k .

We also put $A(\sigma) = L(\sigma) + \partial B(\sigma)$ and use (2.2) in the following form:

$$(2.3) \quad \chi_C(A(\partial\tau)) = 0.$$

Let us check that the $(m+1)$ -skeleton Q_m of Q is already balanced (so that we may put $B(\cdot) = 0$). Let $|C| = m+1$, then

$$\chi_C(L(\partial\tau)) = (\delta\chi_C, L(\tau)) = 0,$$

since $\delta\chi_C = 0$ by the formula (2.1) and the assumption of the theorem (no face of T is colored in $m+2$ different colors).

The plan of the remaining part of the proof is following:

- Denote the k -skeleton of any facet of the cube corresponding to Q_k by Q_{k-1} ;
- Balance Q_{k-1} by defining a suitable $B: C_{k-1}(Q_{k-1}) \rightarrow C_k(T)$;
- Make sure that in all the expressions $B(\sigma)$ for all $(k-1)$ -faces $\sigma \in Q_{k-1}$ every k -face $\beta \in T$ is used at most $C(d, k-1)$ times (counted with its multiplicity in the chains $B(\sigma)$).

If this plan passes then on the last stage we have a 1-dimensional skeleton of a $(d-m)$ -dimensional cube (containing n^{d-m} small cubes) Q_0 . To every vertex $v \in Q_0$ we assign a chain of 1-faces of T denoted by $B(v)$. Then 0-chains $A(v)$ are simply sets of vertices of T with integer multiplicities such that the sum of coefficients in every $A(v)$ is 1. For any color c by (2.3) we obtain that $\chi_c(A(v_1)) = \chi_c(A(v_2))$ for any pair of adjacent vertices v_1 and v_2 . Since Q_0 is a connected graph we obtain that the number $\chi_c(A(v)) = x_c$ does not depend on v . The sum over all colors is

$$\sum \chi_c(A(v)) = (1, A(v)) = 1,$$

so there exists a color c with nonzero x_c . Hence this color is used in every support of the 0-cycle $A(v)$ for every v . We have at least n^{d-m} different choices of v and any point colored in c is counted at most $C'C(d, 0)$ times (here C' is the maximal number of 1-faces incident to a vertex in T).

So it remains to pass from the balancing of Q_k to the balancing of Q_{k-1} . Note that for every k -face $\tau \in Q_{k-1}$ we have to satisfy the equality (since $\partial^2 = 0$ and $\partial A(\tau) = \partial L(\tau) = L(\partial\tau)$):

$$(2.4) \quad \chi_C(\partial A(\tau) + \partial B(\partial\tau)) = \chi_C(L(\partial\tau) + \partial B(\partial\tau)) = 0.$$

In this formula $A(\tau)$ is already defined, and B is to be defined on $(k-1)$ -faces of Q_{k-1} . The equality (2.4) follows from the equality:

$$(2.5) \quad \chi_D(A(\tau) + B(\partial\tau)) = 0$$

for every k -face $\tau \in Q_{k-1}$ and every set D of $k+1$ colors. Indeed, using (2.1) from (2.5) we obtain:

$$(2.6) \quad \begin{aligned} \chi_C(\partial A(\tau) + \partial B(\partial\tau)) &= (\delta\chi_C, A(\tau) + B(\partial\tau)) = \\ &= \left(\sum_{c_k} \chi_{C_{c_k}}, A(\tau) + B(\partial\tau) \right) = 0. \end{aligned}$$

Now we fix a set D of $k+1$ colors. Define by

$$(2.7) \quad \xi_D(\tau) = \chi_D(A(\tau))$$

a k -cocycle on Q_k since for every $(k+1)$ -face $\rho \in Q_k$ we have:

$$(2.8) \quad \xi_D(\partial\rho) = \chi_D(L(\partial\rho) + \partial B\partial\rho) = 0$$

because Q_k is balanced. To make this cocycle zero (as required in (2.5)) we have to assign to some $(k-1)$ -faces $\sigma \in Q_{k-1}$ as $B(\sigma)$ some sets (with coefficients) of k -faces $\tau \in T$. Obviously, it suffices to use only those k -faces $\tau \in T$ that are colored exactly in the colors of D .

The map $\sigma \mapsto \xi_D(B(\sigma))$ is going to be a $(k-1)$ -dimensional cochain in $C^{k-1}(Q_{k-1})$ with coboundary $\xi_D|_{Q_{k-1}}$. In order to use any k -face (out of those colored in D) at most $C(d, k-1)$ times we have to check that the ratio between the norm (sum of absolute values) of some $(k-1)$ -cochain $\eta \in C^{k-1}(Q_{k-1})$ such that $\delta\eta = \xi_D$ on Q_{k-1} and the number of k -faces usable in $B(\sigma)$ (that is, colored in D) is bounded by a constant $C(d, k-1)$.

Let the norm of ξ_D as an element of $C^k(Q_k)$ equal M . By the assumption that in every $B(\tau)$ a $(k+1)$ -face of T is used at most $C(d, k)$ times we conclude that every k -face of T is used in all $A(\tau)$ at most $C(d, k)C'(k)$ times, where $C'(k)$ is the maximal number of $(k+1)$ -faces containing a given k -face of T (it can be bounded independently on the choice of a particular triangulation T). By the formula $\xi_D(\tau) = \chi_D(A(\tau))$ we conclude that among the k -faces of T there do exist at least $\frac{M}{C(d, k)C'(k)}$ “candidates” for $B(\sigma)$. Now it suffices to solve the equation $\delta\eta = \xi_D$ on cochains on Q_{k-1} so that the norm $|\eta|$ is at most $C''(d, k-1)M$. After that we can assign to cubical faces of Q_{k-1} on which η is nonzero several faces of T on which ξ_D is nonzero.

Note that $|\xi_D| = M$ and for some codimension 1 cubical section Q' of Q_k (parallel to Q_{k-1}) we have: $|\xi_D|_{Q'} \leq M/n$. Then we use the “filling inequality” (see for example [2], where filling inequalities are widely used):

Lemma 2.2. *For a k -dimensional cocycle α on the cubical partition of the d' -dimensional cube Q' (in terms of this proof) there exists a $(k-1)$ -dimensional cubical cochain β such that $\delta\beta = \alpha$ and $|\beta| \leq C_F(d', k)n|\alpha|$.*

By this lemma we select a $(k - 1)$ -dimensional cochain β on Q' with norm at most $MC_F(d - m + k, k)$ with coboundary $\xi_D|_{Q'}$. Denote the part of ξ_D between Q' and Q_{k-1} by ξ'_D ; this is a cochain with norm at most M . As the required $(k - 1)$ -dimensional cochain η on Q_{k-1} we may take:

$$(2.9) \quad \eta = \beta + \pi_*(\xi'_D)$$

with norm at most $(C_F(d - m + k, k) + 1)M$. Here β is moved from Q' to Q_{k-1} by the translation and by $\pi_*(\xi'_D)$ we mean the *direct image* under the projection onto Q_{k-1} that drops the dimension by 1. The cochain $\pi_*(\xi'_D)$ can be defined explicitly (thanks to the cubical complexes that we use) as taking any $(k - 1)$ -face $\sigma \in Q_{k-1}$ to the sum of values (with appropriate signs) of ξ'_D on k -faces of Q_k that project onto σ . In other words:

$$\pi_*(\xi'_D)(\tau) = \xi'_D(\pi^{-1}(\tau)).$$

So we satisfy the equality (2.5) for a particular color set D . Now we can add the chains $B(\sigma)$ corresponding to different sets D . The expressions $B(\sigma)$ for every particular D contained exclusively k -faces colored in the colors in D . Hence for different D we use different faces, which guarantees the bounded multiplicity of the union of all $B(\sigma)$. The faces of $B(\sigma)$ colored in D do not affect the equality (2.5) for another set D' (not obtained from D by a permutation). Now to complete the proof it remains to prove the lemma.

Proof of Lemma 2.2. The proof is similar to the proof of [3, Lemma 2.6], which is stated in terms of cycles Poincaré dual to the cocycles in this proof.

Put $\alpha_0 = \alpha$. We are going to build a cocycle α_{i+1} out of α_i as follows. Take a hyperplane section Z of Q' parallel to a pair of its opposite facets so that $|\alpha_i|_Z| \leq |\alpha_i|/n$. This is possible by the Dirichlet principle.

Define a $(k - 1)$ -cochain β_i as follows:

$$\beta_i(\tau) = \alpha_i([\tau, \pi_Z(\tau)]),$$

where $[\tau, \pi_Z(\tau)]$ is (at most k)-dimensional parallelepiped between τ and its projection onto Z with appropriate sign.

Now we put

$$\alpha_{i+1} = \alpha_i - \delta\beta_i$$

and note that $|\beta_i|$ is at most $n|\alpha_i|$. Note also that α_{i+1} takes the same values as the translation of $\alpha_i|_Z$ on sections parallel to Z and is zero on any face orthogonal to Z .

After several such operations for different directions of Z ($d' - k + 1$ will be enough) the cocycle α_{i+1} becomes zero. We have the inequality:

$$|\alpha_{i+1}| \leq |\alpha_i| \leq \dots \leq |\alpha|.$$

If we take $\beta = \sum_i \beta_i$ then the required inequality holds with constant $C_F(d', k) \leq d' - k + 1$. \square

3. SOME COROLLARIES

We give some topological corollaries of Theorem 1.1:

Corollary 3.1. ¹ *Let a d -dimensional cube Q be covered by closed sets C_i so that no point is covered more than $m + 1$ times. Then one of the sets C_i intersects at least $d - m$ pairs of opposite facets of Q .*

Remark 3.2. This is a generalization of the Lebesgue theorem.

¹The statement of this corollary is suggested by R. Karasev as a simpler version of Corollary 3.3.

Proof. We pass in a standard way from the covering to coloring the vertices of the partition of Q into n^d cubes. If the partition is fine enough then no partition face has $m+2$ distinct colors.

Let us include Q into the cube $2Q$ of size $(2n)^d$ and repeat the coloring of Q using reflections with respect to the halving hyperplanes of $2Q$. Then we extend the coloring onto the whole \mathbb{Z}^d with translations by $\pm 2n$ along the coordinate axes. Let us see what happens with a color c_i . Following Remark 1.3 we assume that the color c_i makes a connected subset of Q . The vertices of \mathbb{Z}^d colored in c_i can be decomposed into connected components; denote one of them by c'_i . If the component c_i spans a pair of opposite facets (orthogonal to a base vector e_j) in Q then c'_i is invariant under the translation by $\pm 2ne_j$. Otherwise c_i does not touch one of the facets orthogonal to e_j and c'_i is trapped between a pair of hyperplanes orthogonal to e_j at distance $2n$ from each other.

So the free Abelian group Λ_i of translational symmetries of c'_i has dimension exactly ℓ , where ℓ is the number of pairs of opposite facets of Q intersected by c_i and c'_i can be obtained from $2Q \cap c'_i$ by translations in Λ_i . If we intersect c'_i with a large cube Q' of size $(2nN)^d$ then the cardinality of $c'_i \cap Q'$ has the growth order N^ℓ for varying N . By Theorem 1.1 and Remark 1.3 some $c'_i \cap Q'$ must have the number of vertices of order at least N^{d-m} ; so for some of c'_i we must have $\ell \geq d-m$. \square

Corollary 3.3. *Let a d -dimensional torus T^d be covered by open sets C_i so that no point is covered more than $m+1$ times. Then for some C_i the image of $H_1(C_i)$ in $H_1(T^d) = \mathbb{Z}^d$ has dimension at least $d-m$.*

Remark 3.4. The sets have to be open so that the connectedness and the arcwise connectedness coincide.

Proof. As in the previous proof we pass from the covering of T^d to a fine enough triangulation of a covering cube Q , which subdivides the cubical partition into n^d small cubes. Then we assume that the vertices of the triangulation are colored so that no face has more than $m+1$ colors. Duplicating Q by translations we obtain a large cube Q_N with side length Nn and the corresponding coloring. By gluing the opposite facets of Q_N we obtain a torus naturally N^d -fold covering T^d .

By Theorem 1.1 in Q_N we have a monochromatic connected component S with size of order N^{d-m} . Let S_1 be the maximal intersection of S with a residue class modulo Q (points are equal modulo Q if the differences of their coordinates are divisible by n). Then $|S_1| \geq |S|/n^d \geq \frac{N^{d-m}}{n^d}$.

Note that a projection of a monochromatic path in S starting in a point of S_1 with coordinates (x_1, \dots, x_d) and ending in a point of S_1 with coordinates (y_1, \dots, y_d) is a monochromatic closed loop in T^d representing the homology class $(\frac{y_i - x_i}{n})_i$. So it suffices to show that the dimension of the linear space generated by pairwise differences of S_1 is at least $d-m$ (the covering set corresponding to S will be the one required). Equivalently, we have to show that the dimension of the affine hull of S_1 is at least $d-m$.

Assume the contrary: The dimension of the affine hull of S_1 is at most $d-m-1$. Then S_1 is contained in at most $(d-m-1)$ -dimensional affine subspace L and the number of vertices in $Q_N \cap L$ (and therefore the number of vertices in S_1) is at most $(Nn)^{d-m-1}$. For large enough N we obtain a contradiction with the inequality $|S_1| \geq \frac{N^{d-m}}{n^d}$. \square

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